

# GEOMETRIC REALIZATIONS OF LUSZTIG'S SYMMETRIES ON THE WHOLE QUANTUM GROUPS

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**ABSTRACT.** In this paper, we shall study the structure of the Grothendieck group of the category consisting of Lusztig's perverse sheaves and give a decomposition theorem of it. By using this decomposition theorem and the geometric realizations of Lusztig's symmetries on the positive part of a quantum group, we shall give geometric realizations of Lusztig's symmetries on the whole quantum group.

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## 1. INTRODUCTION

1.1. Let  $\mathbf{U}$  be the quantum group associated to a Cartan datum, which is introduced by Drinfeld ([4]) and Jimbo ([6]) respectively in the study of quantum Yang-Baxter equations. As a quantization of the universal enveloping algebra of a Kac-Moody Lie algebra, the quantum group  $\mathbf{U}$  is a Hopf algebra and has the following triangular decomposition

$$\mathbf{U} \cong \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+.$$

In [14], Lusztig also introduced an algebra  $\mathbf{f}$  (called the Lusztig's algebra) associated to a Cartan datum, satisfying that there are two monomorphisms of algebras  ${}^+ : \mathbf{f} \rightarrow \mathbf{U}$  and  ${}^- : \mathbf{f} \rightarrow \mathbf{U}$  with images  $\mathbf{U}^+$  and  $\mathbf{U}^-$  respectively.

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Lusztig introduced some operators  $T_i$  on the quantum group  $\mathbf{U}$  satisfying the braid group relations, which are called Lusztig's symmetries ([9, 11]). By the definition of Lusztig's symmetries, the image of  $\mathbf{U}^+$  under  $T_i$  is not contained in  $\mathbf{U}^+$  for any  $i$ . So Lusztig introduced two subalgebras  ${}_i\mathbf{f} = \{x \in \mathbf{f} \mid T_i(x^+) \in \mathbf{U}^+\}$  and  ${}^i\mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x^+) \in \mathbf{U}^+\}$  of  $\mathbf{f}$  ([14]). Note that there exists a unique  $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$  such that  $T_i(x^+) = T_i(x)^+$ .

For any finite quiver  $Q = (I, H)$ , Ringel introduced the Ringel-Hall algebra as an algebraic model of the positive part of the corresponding quantum group ([15]). Green ([5]) introduced the comultiplication on the Ringel-Hall algebra and Xiao ([18]) introduced the antipode. Under these operators, the Ringel-Hall algebra has a Hopf algebra structure. Xiao also considered the Drinfeld double of the Ringel-Hall algebra (called the double Ringel-Hall algebra), the composition subalgebra of which gives a realization of the whole quantum group ([18]).

By this algebraic realization of a quantum group, Ringel applied the BGP reflection functors ([1]) to give realizations of Lusztig's symmetries  $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$  ([16]). Then Xiao-Yang ([20]) and Sevenhant-Van den Bergh ([17]) realized Lusztig's symmetries  $T_i : \mathbf{U} \rightarrow \mathbf{U}$  also by using the BGP reflection functors. Indeed, this method is also available to give precise constructions of Lusztig's symmetries on a double Ringel-Hall algebra ([2, 3]).

1.2. In [10, 12], Lusztig gave a geometric realization of the Lusztig's algebra  $\mathbf{f}$ . Let  $Q = (I, H)$  be the quiver corresponding to  $\mathbf{f}$ . Inspired by the algebraic realization of  $\mathbf{f}$  given by Ringel, Lusztig considered the variety  $E_{\mathbf{V}}$  consisting of representations with dimension vector  $\nu$  of the quiver  $Q$  and the category  $\mathcal{Q}_{\mathbf{V}}$  of some semisimple complexes on  $E_{\mathbf{V}}$ . Let  $K(\mathcal{Q}_{\mathbf{V}})$  be the Grothendieck group of  $\mathcal{Q}_{\mathbf{V}}$ . Considering all dimension vectors, let

$$K(\mathcal{Q}) = \bigoplus_{\nu} K(\mathcal{Q}_{\mathbf{V}}),$$

which is isomorphic to the Lusztig's algebra  $\mathbf{f}$ .

In [19], Xiao, Xu and Zhao considered a larger category of Weil complexes on the variety  $E_{\mathbf{V}}$  for any dimension vector  $\nu$ . They showed that the direct sum of the Grothendieck groups of these categories gives a realization of the generic Ringel-Hall algebra via Lusztig's geometric method. They also considered the Drinfeld double of the direct sum and gave a realization of the generic double Ringel-Hall algebra.

By using the method of Lusztig, Kato gave geometric realizations of Lusztig's symmetries  $T_i : {}_i\mathbf{f} \rightarrow {}^i\mathbf{f}$  in the case of finite type for all  $i$  ([7]). Then his constructions were generalized by Xiao and Zhao to all symmetrizable Cartan datum ([21, 22]).

Assume that  $i$  is a sink (resp. source) of the quiver  $Q$ . Xiao and Zhao considered a subvariety  ${}_iE_{\mathbf{V},0}$  (resp.  ${}^iE_{\mathbf{V},0}$ ) of  $E_{\mathbf{V}}$  and a category  ${}_i\mathcal{Q}_{\mathbf{V},0}$  (resp.  ${}^i\mathcal{Q}_{\mathbf{V},0}$ ) of some semisimple complexes on  ${}_iE_{\mathbf{V},0}$  (resp.  ${}^iE_{\mathbf{V},0}$ ). They showed that  $K({}_i\mathcal{Q}_0) = \bigoplus_{\nu} K({}_i\mathcal{Q}_{\mathbf{V},0})$  (resp.  $K({}^i\mathcal{Q}_0) = \bigoplus_{\nu} K({}^i\mathcal{Q}_{\mathbf{V},0})$ ) realizes  ${}_i\mathbf{f}$  (resp.  ${}^i\mathbf{f}$ ).

Let  $i \in I$  be a sink of the quiver  $Q$ . Then  $i$  is a source of  $Q' = \sigma_i Q$ , which is the quiver by reversing the directions of all arrows in  $Q$  containing  $i$ . They defined a map  $\tilde{\omega}_i : K({}_i \mathcal{Q}_0) \rightarrow K({}^i \mathcal{Q}_0)$ , which gives a realization of Lusztig's symmetry  $T_i : {}_i \mathbf{f} \rightarrow {}^i \mathbf{f}$ .

1.3. In this paper, we shall give a geometric realization of Lusztig's symmetry  $T_i : \mathbf{U} \rightarrow \mathbf{U}$  for any  $i$ .

Let  $Q$  be the quiver corresponding to  $\mathbf{U}$ . First, we shall construct a skew-Hopf pairing  $(\tilde{K}(\mathcal{Q})^+, \tilde{K}(\mathcal{Q})^-, \varphi)$ , where  $\tilde{K}(\mathcal{Q})^+$  and  $\tilde{K}(\mathcal{Q})^-$  are two Hopf algebras by adding torus algebra  $\mathbf{K} = \bigoplus_{\mu} \mathbf{A} \mathbf{k}_{\mu}$  to  $K(\mathcal{Q})$ . Let  $DK(\mathcal{Q}) = DK(\mathcal{Q})(Q)$  be the quotient of the Drinfeld double of this skew-Hopf pairing module the Hopf ideal generated by  $\mathbf{k}_{\mu} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{k}_{\mu}$ . Then  $DK(\mathcal{Q})$  is isomorphic to the whole quantum group  $\mathbf{U}$  and has the following triangular decomposition

$$DK(\mathcal{Q}) \cong K(\mathcal{Q})^- \otimes \mathbf{K} \otimes K(\mathcal{Q})^+.$$

Then, we shall study the structure of  $K(\mathcal{Q})$ . Assume that  $i$  is a sink of the quiver  $Q$ . We have the following theorem.

**Theorem 1.** *The  $\mathcal{A}$ -module  $K(\mathcal{Q})$  has the following direct sum decomposition*

$$K(\mathcal{Q}) \cong \bigoplus_{r \geq 0} [\mathcal{L}_{ri}] * (i j_0)! (K({}_i \mathcal{Q}_0)).$$

This theorem is a geometric interpretation of the following direct sum decomposition

$$\mathbf{f} = \bigoplus_{t \geq 0} \theta_i^{(t)} {}_i \mathbf{f}.$$

When  $i$  is a source, we have a similar result.

Assume that  $i$  is also a sink of the quiver  $Q$ . So  $i$  is a source of  $Q' = \sigma_i Q$ . By using Theorem 1 and the map  $\tilde{\omega}_i : K({}_i \mathcal{Q}_0) \rightarrow K({}^i \mathcal{Q}_0)$ , we can define a map

$$\tilde{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q').$$

Then we have the following main theorem in this paper.

**Theorem 2.** *The map  $\tilde{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q')$  is an isomorphism of Hopf algebras satisfying that the following diagram is commutative*

$$\begin{array}{ccc} DK(\mathcal{Q})(Q) & \xrightarrow{\tilde{T}_i} & DK(\mathcal{Q})(Q') \\ \downarrow \cong & & \downarrow \cong \\ \mathbf{U}_{\mathcal{A}} & \xrightarrow{T_i} & \mathbf{U}_{\mathcal{A}}. \end{array}$$

## 2. QUANTUM GROUPS AND LUSZTIG'S SYMMETRIES

In this section, we shall recall the definitions of quantum groups and Lusztig's symmetries. We shall follow the notations in [14].

2.1. Let  $I$  be a finite index set with  $|I| = n$ ,  $A = (a_{ij})_{i,j \in I}$  be a symmetric generalized Cartan matrix, and  $(A, \Pi, \Pi^\vee, P, P^\vee)$  be a Cartan datum associated with  $A$ , where  $\Pi = \{\alpha_i \mid i \in I\}$  is the set of simple roots,  $\Pi^\vee = \{h_i \mid i \in I\}$  is the set of simple coroots,  $P$  is the weight lattice and  $P^\vee$  is the dual weight lattice. There is a symmetric bilinear form  $(-, -)$  on  $\mathbb{Z}I$  induced by  $(i, j) = \alpha_j(h_i) = a_{ij}$ . In this paper, assume that  $P^\vee = \mathbb{Z}\Pi^\vee$  and the symmetric bilinear form on  $P^\vee$  induced by  $(h_i, h_j) = (i, j)$  is also denoted by  $(-, -)$ .

The quantum group  $\mathbf{U}$  associated with a Cartan datum  $(A, \Pi, \Pi^\vee, P, P^\vee)$  is an associative algebra over  $\mathbb{Q}(v)$  with unit element  $\mathbf{1}$ , generated by the elements  $E_i$ ,  $F_i (i \in I)$  and  $K_\mu (\mu \in P^\vee)$  subject to the following relations

- (1)  $K_0 = \mathbf{1}$  and  $K_\mu K_{\mu'} = K_{\mu+\mu'}$  for all  $\mu, \mu' \in P^\vee$ ,
- (2)  $K_\mu E_i K_{-\mu} = v^{\alpha_i(\mu)} E_i$  for all  $i \in I$ ,  $\mu \in P^\vee$ ,
- (3)  $K_\mu F_i K_{-\mu} = v^{-\alpha_i(\mu)} F_i$  for all  $i \in I$ ,  $\mu \in P^\vee$ ,
- (4)  $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_{-i}}{v - v^{-1}}$  for all  $i, j \in I$ ,
- (5)  $\sum_{k=0}^{1-a_{ij}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{ij}-k)} = 0$  for all  $i \neq j \in I$ ,
- (6)  $\sum_{k=0}^{1-a_{ij}} (-1)^k F_j F_i^{(k)} F_i^{(1-a_{ij}-k)} = 0$  for all  $i \neq j \in I$ ,

where  $K_i = K_{h_i}$ ,  $[n]_v = \frac{v^n - v^{-n}}{v - v^{-1}}$ ,  $E_i^{(n)} = E_i^n / [n]_v!$  and  $F_i^{(n)} = F_i^n / [n]_v!$ .

The comultiplication  $\Delta : \mathbf{U} \rightarrow \mathbf{U} \otimes \mathbf{U}$  is an algebra homomorphism satisfying that

- (1)  $\Delta(E_i) = E_i \otimes \mathbf{1} + K_i \otimes E_i$  for all  $i \in I$ ,
- (2)  $\Delta(F_i) = F_i \otimes K_{-i} + \mathbf{1} \otimes F_i$  for all  $i \in I$ ,
- (3)  $\Delta(K_\mu) = K_\mu \otimes K_\mu$  for all  $\mu \in P^\vee$ .

The antipode  $S : \mathbf{U} \rightarrow \mathbf{U}^{op}$  is an algebra homomorphism satisfying that

- (1)  $S(E_i) = -K_{-i} E_i$  for all  $i \in I$ ,
- (2)  $S(F_i) = -F_i K_{-i}$  for all  $i \in I$ ,
- (3)  $S(K_\mu) = K_{-\mu}$  for all  $\mu \in P^\vee$ .

The counit  $\mathbf{e} : \mathbf{U} \rightarrow \mathbb{Q}(v)$  is also an algebra homomorphism satisfying that

- (1)  $\mathbf{e}(E_i) = 0$  for all  $i \in I$ ,
- (2)  $\mathbf{e}(F_i) = 0$  for all  $i \in I$ ,
- (3)  $\mathbf{e}(K_\mu) = 1$  for all  $\mu \in P^\vee$ .

**Theorem 2.1** ([14]). *The algebra  $(\mathbf{U}, \Delta, S, \mathbf{e})$  is a Hopf algebra.*

The quantum group  $\mathbf{U}$  has the following triangular decomposition

$$\mathbf{U} \cong \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+,$$

where  $\mathbf{U}^-$ ,  $\mathbf{U}^+$  and  $\mathbf{U}^0$  are the subalgebras  $\mathbf{U}$  generated by  $F_i$ ,  $E_i$  and  $K_\mu$  for all  $i \in I$  and  $\mu \in P^\vee$  respectively.

2.2. In [14], Lusztig also introduced an associative  $\mathbb{Q}(v)$ -algebra  $\mathbf{f}$ , which is generated by  $\theta_i (i \in I)$  subject to the quantum Serre relations  $\sum_{k=0}^{1-a_{ij}} (-1)^k \theta_i^{(k)} \theta_j \theta_i^{(1-a_{ij}-k)} = 0$  for all  $i \neq j \in I$ , where  $\theta_i^{(n)} = \theta_i^n / [n]_v!$ . There are two well-defined  $\mathbb{Q}(v)$ -algebra

monomorphisms  $^+ : \mathbf{f} \rightarrow \mathbf{U}$  and  $^- : \mathbf{f} \rightarrow \mathbf{U}$  satisfying  $E_i = \theta_i^+$  and  $F_i = \theta_i^-$  for all  $i \in I$  and the images of  $^+$  and  $^-$  are  $\mathbf{U}^+$  and  $\mathbf{U}^-$  respectively.

Define  $|\theta_i| = i \in \mathbb{N}I$  for any  $i \in I$  and  $|xy| = |x| + |y|$  by induction. There exists a unique algebra homomorphism  $r : \mathbf{f} \rightarrow \mathbf{f} \otimes \mathbf{f}$  such that  $r(\theta_i) = \theta_i \otimes \mathbf{1} + \mathbf{1} \otimes \theta_i$  for any  $i \in I$ , where the multiplication on  $\mathbf{f} \otimes \mathbf{f}$  is defined as  $(x \otimes y)(x' \otimes y') = v^{(|y|, |x'|)} xx' \otimes yy'$ .

2.3. Denote by  $T_i : \mathbf{U} \rightarrow \mathbf{U}$  the Lusztig's symmetries for all  $i \in I$  ([9, 11, 14]). The formulas of  $T_i$  on the generators are

- (1)  $T_i(E_i) = -F_i K_i$  and  $T_i(F_i) = -K_{-i} E_i$ ,
- (2)  $T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r v^{-r} E_i^{(s)} E_j E_i^{(r)}$  for  $i \neq j \in I$ ,
- (3)  $T_i(F_j) = \sum_{r+s=-a_{ij}} (-1)^r v^r F_i^{(r)} F_j F_i^{(s)}$  for  $i \neq j \in I$ ,
- (4)  $T_i(K_\mu) = K_{\mu - \alpha_i(\mu)h_i}$  for all  $\mu \in P^\vee$ .

Let  ${}^i \mathbf{f} = \{x \in \mathbf{f} \mid T_i(x^+) \in \mathbf{U}^+\}$  and  ${}^i \mathbf{f} = \{x \in \mathbf{f} \mid T_i^{-1}(x^+) \in \mathbf{U}^+\}$ , which are subalgebras of  $\mathbf{f}$  ([14]). By the definitions, there exists a unique  $T_i : {}^i \mathbf{f} \rightarrow {}^i \mathbf{f}$  such that  $T_i(x^+) = T_i(x)^+$ .

**Theorem 2.2** ([14]). *The algebra  $\mathbf{f}$  has the following direct sum decompositions*

$$\mathbf{f} = \bigoplus_{t \geq 0} \theta_i^{(t)} {}^i \mathbf{f} = \bigoplus_{t \geq 0} {}^i \mathbf{f} \theta_i^{(t)}.$$

### 3. GEOMETRIC REALIZATION OF THE LUSZTIG'S ALGEBRA $\mathbf{f}$

In this section, we shall recall the geometric realization of the algebra  $\mathbf{f}$  given by Lusztig ([10, 12, 14]).

3.1. A quiver  $Q = (I, H, s, t)$  consists of a vertex set  $I$ , an arrow set  $H$ , and two maps  $s, t : H \rightarrow I$  such that an arrow  $\rho \in H$  starts at  $s(\rho)$  and terminates at  $t(\rho)$ . Let  $h_{ij} = \#\{i \rightarrow j\}$ ,  $a_{ij} = h_{ij} + h_{ji}$  and  $\mathbf{f}$  be the Lusztig's algebra corresponding to  $A = (a_{ij})$ . Let  $p$  be a prime and  $q$  be a power of  $p$ . Denote by  $\mathbb{F}_q$  the finite field with  $q$  elements and  $\mathbb{K} = \overline{\mathbb{F}_q}$ .

For a finite dimensional  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V} = \bigoplus_{i \in I} V_i$ , define

$$E_{\mathbf{V}} = E_{\mathbf{V}, Q} = \bigoplus_{\rho \in H} \text{Hom}_{\mathbb{K}}(V_{s(\rho)}, V_{t(\rho)}).$$

The dimension vector of  $\mathbf{V}$  is defined as  $\underline{\dim} \mathbf{V} = \sum_{i \in I} (\dim_{\mathbb{K}} V_i) i \in \mathbb{N}I$ , which can also be viewed as an element in the weight lattice  $P$ . The algebraic group  $G_{\mathbf{V}} = \prod_{i \in I} GL_{\mathbb{K}}(V_i)$  acts on  $E_{\mathbf{V}}$  naturally.

Fix a nonzero element  $\nu \in \mathbb{N}I$ . Let

$$Y_{\nu} = \{\mathbf{y} = (\mathbf{i}, \mathbf{a}) \mid \sum_{l=1}^k a_l i_l = \nu\},$$

where  $\mathbf{i} = (i_1, i_2, \dots, i_k)$ ,  $i_l \in I$ ,  $\mathbf{a} = (a_1, a_2, \dots, a_k)$ ,  $a_l \in \mathbb{N}$ . Fix a finite dimensional  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  such that  $\underline{\dim} \mathbf{V} = \nu$ . For any element  $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ , a flag

of type  $\mathbf{y}$  in  $\mathbf{V}$  is a sequence  $\phi = (\mathbf{V} = \mathbf{V}^k \supset \mathbf{V}^{k-1} \supset \dots \supset \mathbf{V}^0 = 0)$  of  $I$ -graded  $\mathbb{K}$ -vector spaces such that  $\underline{\dim} \mathbf{V}^l / \mathbf{V}^{l-1} = a_l i_l$ .

Let  $F_{\mathbf{y}}$  be the variety of all flags of type  $\mathbf{y}$  in  $\mathbf{V}$ . For any  $x = (x_{\rho})_{\rho \in H} \in E_{\mathbf{V}}$ , a flag  $\phi$  is called  $x$ -stable if  $x_{\rho}(V_{s(\rho)}^l) \subset V_{t(\rho)}^l$  for all  $l$  and all  $\rho \in H$ . Let

$$\tilde{F}_{\mathbf{y}} = \{(x, \phi) \in E_{\mathbf{V}} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and  $\pi_{\mathbf{y}} : \tilde{F}_{\mathbf{y}} \rightarrow E_{\mathbf{V}}$  be the projection to  $E_{\mathbf{V}}$ .

Let  $\overline{\mathbb{Q}}_l$  be the  $l$ -adic field and  $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$  be the bounded  $G_{\mathbf{V}}$ -equivariant derived category of  $\overline{\mathbb{Q}}_l$ -constructible complexes on  $E_{\mathbf{V}}$ . For each  $\mathbf{y} \in Y_{\nu}$ ,  $\mathcal{L}_{\mathbf{y}} = (\pi_{\mathbf{y}})_!(1_{\tilde{F}_{\mathbf{y}}})[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$  is a semisimple perverse sheaf, where  $d_{\mathbf{y}} = \dim \tilde{F}_{\mathbf{y}}$  and  $\mathcal{L}(d)$  is the Tate twist of  $\mathcal{L}$ . Let  $\mathcal{P}_{\mathbf{V}}$  be the set of simple perverse sheaves  $\mathcal{L}$  on  $E_{\mathbf{V}}$  such that  $\mathcal{L}[r]$  appears as a direct summand of  $\mathcal{L}_{\mathbf{y}}$  for some  $\mathbf{y} \in Y_{\nu}$  and  $r \in \mathbb{Z}$ . Let  $\mathcal{Q}_{\mathbf{V}}$  be the full subcategory of  $\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$  consisting of all complexes which are isomorphic to finite direct sums of complexes in the set  $\{\mathcal{L}[r] \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}, r \in \mathbb{Z}\}$ .

Let  $K(\mathcal{Q}_{\mathbf{V}})$  be the Grothendieck group of  $\mathcal{Q}_{\mathbf{V}}$  and define  $v^{\pm}[\mathcal{L}] = [\mathcal{L}[\pm 1](\pm \frac{1}{2})]$  for any  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$ . Then,  $K(\mathcal{Q}_{\mathbf{V}})$  is a free  $\mathcal{A}$ -module, where  $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ . Define

$$K(\mathcal{Q}) = \bigoplus_{\nu \in \mathbb{N}I} K(\mathcal{Q}_{\mathbf{V}}).$$

Let  $\mathbf{B}_{\nu} = \{[\mathcal{L}] \mid \mathcal{L} \in \mathcal{P}_{\mathbf{V}}\}$  and  $\mathbf{B} = \sqcup_{\nu \in \mathbb{N}I} \mathbf{B}_{\nu}$ . Then  $\mathbf{B}$  is the canonical basis of  $K(\mathcal{Q})$  introduced by Lusztig in [10, 12].

3.2. For  $\nu, \nu', \nu'' \in \mathbb{N}I$  such that  $\nu = \nu' + \nu''$  and three  $I$ -graded  $\mathbb{K}$ -vector spaces  $\mathbf{V}, \mathbf{V}', \mathbf{V}''$  such that  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}' = \nu'$ ,  $\underline{\dim} \mathbf{V}'' = \nu''$ , Lusztig introduced the induction functor

$$\text{Ind}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}} : \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''} \rightarrow \mathcal{Q}_{\mathbf{V}},$$

which induces the following  $\mathcal{A}$ -bilinear operator

$$\begin{aligned} * = \text{ind}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}} : K(\mathcal{Q}_{\mathbf{V}'} ) \times K(\mathcal{Q}_{\mathbf{V}''} ) &\rightarrow K(\mathcal{Q}_{\mathbf{V}}) \\ ([\mathcal{L}'], [\mathcal{L}'']) &\mapsto [\mathcal{L}'] * [\mathcal{L}''] = [\text{Ind}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}}(\mathcal{L}', \mathcal{L}'')]. \end{aligned}$$

Under these operators,  $K(\mathcal{Q})$  becomes an associative  $\mathcal{A}$ -algebra.

For  $\nu, \nu_1, \nu_2, \dots, \nu_s \in \mathbb{N}I$  such that  $\nu = \nu_1 + \nu_2 + \dots + \nu_s$  and  $I$ -graded  $\mathbb{K}$ -vector spaces  $\mathbf{V}, \mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_s$  such that  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}_l = \nu_l$ , we can define the  $s$ -fold version of the induction functor  $\text{ind}_{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_s}^{\mathbf{V}}$  by induction ([19]).

3.3. For  $\nu, \nu', \nu'' \in \mathbb{N}I$  such that  $\nu = \nu' + \nu''$  and three  $I$ -graded  $\mathbb{K}$ -vector spaces  $\mathbf{V}, \mathbf{V}', \mathbf{V}''$  such that  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}' = \nu'$ ,  $\underline{\dim} \mathbf{V}'' = \nu''$ , Lusztig introduced the restriction functor

$$\text{Res}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}} : \mathcal{Q}_{\mathbf{V}} \rightarrow \mathcal{Q}_{\mathbf{V}'} \times \mathcal{Q}_{\mathbf{V}''},$$

which induces the following  $\mathcal{A}$ -bilinear operator

$$\begin{aligned} \text{res}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}} : K(\mathcal{Q}_{\mathbf{V}}) &\rightarrow K(\mathcal{Q}_{\mathbf{V}'} ) \times K(\mathcal{Q}_{\mathbf{V}''} ) \\ [\mathcal{L}] &\mapsto [\text{Res}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}}(\mathcal{L})]. \end{aligned}$$

Under these operators, we have an operator  $\text{res} : K(\mathcal{Q}) \rightarrow K(\mathcal{Q}) \otimes K(\mathcal{Q})$ .

For  $\nu, \nu_1, \nu_2, \dots, \nu_s \in \mathbb{N}I$  such that  $\nu = \nu_1 + \nu_2 + \dots + \nu_s$  and  $I$ -graded  $\mathbb{K}$ -vector spaces  $\mathbf{V}, \mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_s$  such that  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}_i = \nu_i$ , we can define the  $s$ -fold version of the restriction functor  $\text{res}_{\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_s}^{\mathbf{V}}$  by induction ([19]).

3.4. Fix  $\nu, \nu_1, \nu_2, \nu', \nu'' \in \mathbb{N}I$  such that  $\nu = \nu_1 + \nu_2 = \nu' + \nu''$  and  $I$ -graded  $\mathbb{K}$ -vector spaces  $\mathbf{V}, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}', \mathbf{V}''$  such that  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}_1 = \nu_1$ ,  $\underline{\dim} \mathbf{V}_2 = \nu_2$ ,  $\underline{\dim} \mathbf{V}' = \nu'$ ,  $\underline{\dim} \mathbf{V}'' = \nu''$ .

**Proposition 3.1** ([12, 14, 19]). *For any  $\mathcal{L}_1 \in \mathcal{Q}_{\mathbf{V}_1}$  and  $\mathcal{L}_2 \in \mathcal{Q}_{\mathbf{V}_2}$ , we have*

$$\text{Res}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}}(\text{Ind}_{\mathbf{V}_1, \mathbf{V}_2}^{\mathbf{V}}(\mathcal{L}_1, \mathcal{L}_2)) = \bigoplus_{\nu'_1, \nu''_1, \nu'_2, \nu''_2} \widehat{\text{Ind}_{\mathbf{V}'_1, \mathbf{V}'_2}^{\mathbf{V}'} \otimes \text{Ind}_{\mathbf{V}''_1, \mathbf{V}''_2}^{\mathbf{V}''}}(\text{Res}_{\mathbf{V}'_1, \mathbf{V}'_1}^{\mathbf{V}_1}(\mathcal{L}_1), \text{Res}_{\mathbf{V}''_2, \mathbf{V}''_2}^{\mathbf{V}_2}(\mathcal{L}_2)),$$

where  $\mathbf{V}'_1, \mathbf{V}''_1, \mathbf{V}'_2, \mathbf{V}''_2$  are  $I$ -graded  $\mathbb{K}$ -vector spaces with dimension vectors  $\nu'_1, \nu''_1, \nu'_2, \nu''_2$  such that  $\nu'_1 + \nu''_1 = \nu_1$ ,  $\nu'_2 + \nu''_2 = \nu_2$ ,  $\nu'_1 + \nu'_2 = \nu'$ ,  $\nu''_1 + \nu''_2 = \nu''$  and the functor  $\widehat{\text{Ind}_{\mathbf{V}'_1, \mathbf{V}'_2}^{\mathbf{V}'} \otimes \text{Ind}_{\mathbf{V}''_1, \mathbf{V}''_2}^{\mathbf{V}''}}$  is just the twist of  $\text{Ind}_{\mathbf{V}'_1, \mathbf{V}'_2}^{\mathbf{V}'} \otimes \text{Ind}_{\mathbf{V}''_1, \mathbf{V}''_2}^{\mathbf{V}''}$ .

As a corollary, we have

**Corollary 3.2.** *The operator  $\text{res} : K(\mathcal{Q}) \rightarrow K(\mathcal{Q}) \otimes K(\mathcal{Q})$  is an algebra homomorphism with respect to the twisted multiplication on  $K(\mathcal{Q}) \otimes K(\mathcal{Q})$ .*

**Theorem 3.3** ([12, 14]). *There is a unique  $\mathcal{A}$ -algebra isomorphism*

$$\lambda_{\mathcal{A}} : K(\mathcal{Q}) \rightarrow \mathbf{f}_{\mathcal{A}}$$

such that

$$\lambda_{\mathcal{A}} \otimes \lambda_{\mathcal{A}}(\text{res}(x)) = r(\lambda_{\mathcal{A}}(x))$$

and  $\lambda_{\mathcal{A}}(\mathcal{L}_{\mathbf{y}}) = \theta_{\mathbf{y}}$  for all  $\mathbf{y} \in Y_{\nu}$ , where  $\theta_{\mathbf{y}} = \theta_{i_1}^{(a_1)} \theta_{i_2}^{(a_2)} \dots \theta_{i_k}^{(a_k)}$  and  $\mathbf{f}_{\mathcal{A}}$  is the integral form of  $\mathbf{f}$ .

#### 4. GEOMETRIC REALIZATION OF THE QUANTUM GROUP $\mathbf{U}$

In this section, we shall define a skew-Hopf pairing and show that a quotient of the Drinfeld double of this skew-Hopf pairing is isomorphic to the quantum group  $\mathbf{U}$ .

Let  $Q$  be a quiver and fix a Cartan datum  $(A, \Pi, \Pi^{\vee}, P, P^{\vee})$  with  $A = (a_{ij})$ , where  $a_{ij} = \#\{i \rightarrow j\} + \#\{j \rightarrow i\}$ . Let  $\mathbf{U}$  be the quantum group corresponding to this Cartan datum.

Let  $\mathbf{K} = \bigoplus_{\mu \in P^{\vee}} \mathcal{A} \mathbf{k}_{\mu}$  be the torus algebra and set  $\tilde{K}(\mathcal{Q})^+$  be the free  $\mathcal{A}$ -module with the basis  $\{\mathbf{k}_{\mu}[\mathcal{L}]^+ \mid \mu \in P^{\vee}, [\mathcal{L}] \in \mathbf{B}\}$ . The Hopf algebra structure of  $\tilde{K}(\mathcal{Q})^+$  is given by the following operations ([18, 19]).

(a) The multiplication is defined as

- (1)  $[\mathcal{L}_1]^+ [\mathcal{L}_2]^+ = [\mathcal{L}_1 * \mathcal{L}_2]^+$  for  $\mathcal{L}_i \in \mathcal{Q}_{\mathbf{V}_i}$ ,
- (2)  $\mathbf{k}_{\mu}[\mathcal{L}]^+ \mathbf{k}_{-\mu} = v^{\alpha(\mu)} [\mathcal{L}]^+$  for  $\mu \in P^{\vee}$  and  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$  with  $\underline{\dim} \mathbf{V} = \alpha$ ,
- (3)  $\mathbf{k}_{\mu} \mathbf{k}_{\mu'} = \mathbf{k}_{\mu + \mu'}$  for all  $\mu, \mu' \in P^{\vee}$ .



- (b) The comultiplication is defined as
- (1)  $\tilde{\Delta}([\mathcal{L}]^+) = \sum_{\nu'+\nu''=\nu} \text{res}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}} [\mathcal{L}]^+ (\mathbf{k}_{\mu''} \otimes \mathbf{1})$  for  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$ , where  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}' = \nu'$  and  $\underline{\dim} \mathbf{V}'' = \nu''$ ,
  - (2)  $\tilde{\Delta}(\mathbf{k}_{\mu}) = \mathbf{k}_{\mu} \otimes \mathbf{k}_{\mu}$  for all  $\mu \in P^{\vee}$ .
- (c) The antipode is defined as
- (1)  $\tilde{S}([\mathcal{L}]^+) = \sum_{r \geq 1} (-1)^r \sum_{\nu_1 + \dots + \nu_r = \nu} \mathbf{k}_{-\nu} \text{ind}_{\mathbf{V}_1, \dots, \mathbf{V}_r}^{\mathbf{V}} \text{res}_{\mathbf{V}_1, \dots, \mathbf{V}_r}^{\mathbf{V}} [\mathcal{L}]^+$  for  $0 \not\equiv \mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$ , where  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}_l = \nu_l$ ,
  - (2)  $\tilde{S}(\mathbf{k}_{\mu}) = \mathbf{k}_{-\mu}$  for all  $\mu \in P^{\vee}$ .

Set  $\tilde{K}(\mathcal{Q})^-$  be the free  $\mathcal{A}$ -module with the basis  $\{\mathbf{k}_{\mu}[\mathcal{L}]^- \mid \mu \in P^{\vee}, [\mathcal{L}] \in \mathbf{B}\}$ . The Hopf algebra structure of  $\tilde{K}(\mathcal{Q})^-$  is given by the following operations.

- (a) The multiplication is defined as
- (1)  $[\mathcal{L}_1]^- [\mathcal{L}_2]^- = [\mathcal{L}_1 * \mathcal{L}_2]^-$  for  $\mathcal{L}_i \in \mathcal{Q}_{\mathbf{V}_i}$ ,
  - (2)  $\mathbf{k}_{\mu}[\mathcal{L}]^- \mathbf{k}_{-\mu} = v^{-\alpha(\mu)} [\mathcal{L}]^-$  for  $\mu \in P^{\vee}$  and  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$  with  $\underline{\dim} \mathbf{V} = \alpha$ ,
  - (3)  $\mathbf{k}_{\mu} \mathbf{k}_{\mu'} = \mathbf{k}_{\mu+\mu'}$  for all  $\mu, \mu' \in P^{\vee}$ .
- (b) The comultiplication is defined as
- (1)  $\tilde{\Delta}([\mathcal{L}]^-) = \sum_{\nu'+\nu''=\nu} (\mathbf{1} \otimes \mathbf{k}_{-\mu''}) (\text{res}_{\mathbf{V}', \mathbf{V}''}^{\mathbf{V}})^{op} [\mathcal{L}]^-$  for  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$ , where  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}' = \nu'$  and  $\underline{\dim} \mathbf{V}'' = \nu''$ ,
  - (2)  $\tilde{\Delta}(\mathbf{k}_{\mu}) = \mathbf{k}_{\mu} \otimes \mathbf{k}_{\mu}$  for all  $\mu \in P^{\vee}$ .
- (c) The antipode is defined as
- (1)  $\tilde{S}([\mathcal{L}]^-) = \sum_{r \geq 1} (-1)^r \sum_{\nu_1 + \dots + \nu_r = \nu} \text{ind}_{\mathbf{V}_1, \dots, \mathbf{V}_r}^{\mathbf{V}} (\text{res}_{\mathbf{V}_1, \dots, \mathbf{V}_r}^{\mathbf{V}})^{op} [\mathcal{L}]^- \mathbf{k}_{\nu}$  for  $0 \not\equiv \mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$ , where  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}_l = \nu_l$ ,
  - (2)  $\tilde{S}(\mathbf{k}_{\mu}) = \mathbf{k}_{-\mu}$  for all  $\mu \in P^{\vee}$ .

Fix an  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  with dimension vector  $\nu \in \mathbb{N}I$ . Given  $\mathcal{L}, \mathcal{L}' \in \mathcal{Q}_{\mathbf{V}}$ , let

$$\{\mathcal{L}, \mathcal{L}'\} = \sum_{t \in \mathbb{Z}} \dim H_{G_{\mathbf{V}}}^t(\mathcal{L} \otimes \mathcal{L}', E_{\mathbf{V}}) v^t.$$

This definition can be extended to define a bilinear form  $\varphi : \tilde{K}(\mathcal{Q})^+ \times \tilde{K}(\mathcal{Q})^- \rightarrow \mathbb{Q}(v)$  by setting

$$\varphi(\mathbf{k}_{\alpha}[\mathcal{L}]^+, \mathbf{k}_{\beta}[\mathcal{L}']^-) = v^{-(\alpha, \beta) - \nu(\beta) + \nu'(\alpha)} \{\mathcal{L}, \mathcal{L}'\}$$

for  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$  and  $\mathcal{L}' \in \mathcal{Q}_{\mathbf{V}'}$  such that  $\underline{\dim} \mathbf{V} = \nu$  and  $\underline{\dim} \mathbf{V}' = \nu'$ .

**Proposition 4.1** ([19]). *The triple  $(\tilde{K}(\mathcal{Q})^+, \tilde{K}(\mathcal{Q})^-, \varphi)$  is a skew-Hopf pairing.*

Let  $D(\tilde{K}(\mathcal{Q})^+, \tilde{K}(\mathcal{Q})^-)$  be the Drinfeld double of this skew-Hopf pairing and  $DK(\mathcal{Q}) = DK(\mathcal{Q})(\mathcal{Q})$  be the quotient of  $D(\tilde{K}(\mathcal{Q})^+, \tilde{K}(\mathcal{Q})^-)$  module the Hopf ideal generated by  $\mathbf{k}_{\mu} \otimes \mathbf{1} - \mathbf{1} \otimes \mathbf{k}_{\mu}$  for all  $\mu \in P^{\vee}$ . It is clear that  $DK(\mathcal{Q})$  has the following triangular decomposition

$$(1) \quad DK(\mathcal{Q}) \cong K(\mathcal{Q})^- \otimes \mathbf{K} \otimes K(\mathcal{Q})^+.$$

Theorem 3.3 and the construction of  $DK(\mathcal{Q})$  imply the following theorem.



**Theorem 4.2.** *There exists an isomorphism of Hopf algebras*

$$\lambda_{\mathcal{A}} : DK(\mathcal{Q}) \rightarrow \mathbf{U}_{\mathcal{A}}$$

such that  $\lambda_{\mathcal{A}}([\mathcal{L}_{\mathbf{y}}]^+) = \theta_{\mathbf{y}}^+$ ,  $\lambda_{\mathcal{A}}([\mathcal{L}_{\mathbf{y}}]^-) = \theta_{\mathbf{y}}^-$  and  $\lambda_{\mathcal{A}}(\mathbf{k}_{\mu}) = K_{\mu}$  for all  $\mathbf{y} \in Y_{\nu}$  and  $\mu \in P^{\vee}$ , where  $\mathbf{U}_{\mathcal{A}}$  is the integral form of  $\mathbf{U}$ .

## 5. STRUCTURE OF THE $\mathcal{A}$ -MODULE $K(\mathcal{Q})$

For the geometric definition of Lusztig's symmetries, we shall study the structure of the  $\mathcal{A}$ -module  $K(\mathcal{Q})$  in this section.

5.1. Let  $Q = (I, H, s, t)$  be a quiver. Assume that  $i \in I$  is a sink. Let  $\mathbf{V}$  be a finite dimensional  $I$ -graded  $\mathbb{K}$ -vector space such that  $\underline{\dim} \mathbf{V} = \nu \in \mathbb{N}I$ . For any  $r \in \mathbb{Z}_{\geq 0}$ , consider a subvariety  ${}_iE_{\mathbf{V},r}$  of  $E_{\mathbf{V}}$

$${}_iE_{\mathbf{V},r} = \{x \in E_{\mathbf{V}} \mid \text{Im}(\bigoplus_{\rho \in H, t(\rho)=i} x_{\rho}) \text{ has codimension } r \text{ in } V_i\}.$$

Denote by  ${}_ij_{\mathbf{V},r} : {}_iE_{\mathbf{V},r} \rightarrow E_{\mathbf{V}}$  the natural embedding. Let  $\mathcal{D}_{G_{\mathbf{V}}}^b({}_iE_{\mathbf{V},r})$  be the  $G_{\mathbf{V}}$ -equivariant bounded derived category of  $\overline{\mathbb{Q}}_l$ -constructible complexes on  ${}_iE_{\mathbf{V},r}$ . Naturally, we have two functors  $({}_ij_{\mathbf{V},r})_! : \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V},r}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$  and  ${}_ij_{\mathbf{V},r}^* : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V},r})$ .

Note that the variety  $E_{\mathbf{V}}$  is a disjoint union of  ${}_iE_{\mathbf{V},r}$  for all  $r \geq 0$ . For any  $r \geq 0$ , let  ${}_iE_{\mathbf{V},\geq r} = \bigcup_{r' \geq r} {}_iE_{\mathbf{V},r'}$  and  ${}_iE_{\mathbf{V},\leq r} = \bigcup_{r' \leq r} {}_iE_{\mathbf{V},r'}$ . Let  ${}_i\nu_{\mathbf{V},\geq r} : {}_iE_{\mathbf{V},\geq r} \rightarrow E_{\mathbf{V}}$  be the natural closed embedding and  ${}_ij_{\mathbf{V},\leq r} : {}_iE_{\mathbf{V},\leq r} \rightarrow E_{\mathbf{V}}$  be the natural open embedding.

For any  $\mathbf{y} = (\mathbf{i}, \mathbf{a}) \in Y_{\nu}$ , let

$${}_i\tilde{F}_{\mathbf{y},r} = \{(x, \phi) \in {}_iE_{\mathbf{V},r} \times F_{\mathbf{y}} \mid \phi \text{ is } x\text{-stable}\}$$

and  ${}_i\pi_{\mathbf{y},r} : {}_i\tilde{F}_{\mathbf{y},r} \rightarrow {}_iE_{\mathbf{V},r}$  be the projection to  ${}_iE_{\mathbf{V},r}$ .

For any  $\mathbf{y} \in Y_{\nu}$ ,  ${}_i\mathcal{L}_{\mathbf{y},r} = ({}_i\pi_{\mathbf{y},r})_!(\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y},r}})[d_{\mathbf{y}}](\frac{d_{\mathbf{y}}}{2}) \in \mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V},r})$  is a semisimple perverse sheaf. Let  ${}_i\mathcal{P}_{\mathbf{V},r}$  be the set of simple perverse sheaves  $\mathcal{L}$  on  ${}_iE_{\mathbf{V},r}$  such that  $\mathcal{L}[s]$  appears as a direct summand of  ${}_i\mathcal{L}_{\mathbf{y},r}$  for some  $\mathbf{y} \in Y_{\nu}$  and  $s \in \mathbb{Z}$ . Let  ${}_i\mathcal{Q}_{\mathbf{V},r}$  be the full subcategory of  $\mathcal{D}_{G_{\mathbf{V}}}({}_iE_{\mathbf{V},r})$  consisting of all complexes which are isomorphic to finite direct sums of complexes in the set  $\{\mathcal{L}[s] \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V},r}, s \in \mathbb{Z}\}$ .

Let  $K({}_i\mathcal{Q}_{\mathbf{V},r})$  be the Grothendieck group of  ${}_i\mathcal{Q}_{\mathbf{V},r}$  and define  $v^{\pm}[\mathcal{L}] = [\mathcal{L}[\pm 1]](\pm \frac{1}{2})$  for any  $\mathcal{L} \in {}_i\mathcal{Q}_{\mathbf{V},r}$ . Then,  $K({}_i\mathcal{Q}_{\mathbf{V},r})$  is a free  $\mathcal{A}$ -module. Let

$$K({}_i\mathcal{Q}_r) = \bigoplus_{\nu \in \mathbb{N}I} K({}_i\mathcal{Q}_{\mathbf{V},r}).$$

Let  ${}_i\mathbf{B}_{\mathbf{V},r} = \{[\mathcal{L}] \mid \mathcal{L} \in {}_i\mathcal{P}_{\mathbf{V},r}\}$  and  ${}_i\mathbf{B}_r = \bigcup_{\nu} {}_i\mathbf{B}_{\mathbf{V},r}$ , which is an  $\mathcal{A}$ -basis of  $K({}_i\mathcal{Q}_r)$ .

**Lemma 5.1.** *For any  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  and  $0 \leq r \in \mathbb{Z}$ , we have*

$${}_ij_{\mathbf{V},r}^*(\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V},r}.$$

*Proof.* For any  $\mathbf{y} \in Y_\nu$ , we have the following fiber product

$$\begin{array}{ccc} {}_i\tilde{F}_{\mathbf{y},r} & \xrightarrow{{}_i\tilde{j}_{\mathbf{V},r}} & \tilde{F}_{\mathbf{y}} \\ \downarrow {}_i\pi_{\mathbf{y},r} & & \downarrow \pi_{\mathbf{y}} \\ {}_iE_{\mathbf{V},r} & \xrightarrow{{}_i j_{\mathbf{V},r}} & E_{\mathbf{V}}. \end{array}$$

Hence

$$\begin{aligned} {}_i j_{\mathbf{V},r}^* \mathcal{L}_{\mathbf{y}} &= {}_i j_{\mathbf{V},r}^* (\pi_{\mathbf{y}})_! (\mathbf{1}_{\tilde{F}_{\mathbf{y}}}) [d_{\mathbf{y}}] \left( \frac{d_{\mathbf{y}}}{2} \right) = ({}_i\pi_{\mathbf{y},r})_! {}_i\tilde{j}_{\mathbf{V},r}^* (\mathbf{1}_{\tilde{F}_{\mathbf{y}}}) [d_{\mathbf{y}}] \left( \frac{d_{\mathbf{y}}}{2} \right) \\ &= ({}_i\pi_{\mathbf{y},r})_! (\mathbf{1}_{{}_i\tilde{F}_{\mathbf{y},r}}) [d_{\mathbf{y}}] \left( \frac{d_{\mathbf{y}}}{2} \right) = {}_i\mathcal{L}_{\mathbf{y},r}. \end{aligned}$$

That is  ${}_i j_{\mathbf{V},r}^* (\mathcal{Q}_{\mathbf{V}}) = {}_i\mathcal{Q}_{\mathbf{V},r}$ . □

Hence, we get an  $\mathcal{A}$ -linear map  ${}_i j_r^* : K(\mathcal{Q}) \rightarrow K({}_i\mathcal{Q}_r)$ .

**Theorem 5.2.** *For any  $\mathcal{L} \in \mathcal{Q}_{\mathbf{V}}$  with  $\dim V_i = s$ , let  $\mathcal{L}_{\geq r} = (i_{\mathbf{V},\geq r})_* i_{\mathbf{V},\geq r}^* \mathcal{L}$  and  $\mathcal{L}_r = {}_i j_{\mathbf{V},r}^* \mathcal{L} \in {}_i\mathcal{Q}_{\mathbf{V},r}$ . Then there exists a distinguished triangle*

$$({}_i j_{\mathbf{V},r})_! \mathcal{L}_r \longrightarrow \mathcal{L}_{\geq r} \longrightarrow \mathcal{L}_{\geq r+1} \longrightarrow$$

for any  $0 \leq r \leq s$ .

*Proof.* By Lemma 5.1, we have  $\mathcal{L}_r = {}_i j_{\mathbf{V},r}^* \mathcal{L} \in {}_i\mathcal{Q}_{\mathbf{V},r}$ .

Then, we shall prove this theorem by induction.

By definition,  $\mathcal{L}_{\geq 0} = (i_{\mathbf{V},\geq 0})_* i_{\mathbf{V},\geq 0}^* \mathcal{L} = \mathcal{L}$ . Since  ${}_iE_{\mathbf{V},\geq 1}$  is closed in  $E_{\mathbf{V}}$ , we have the following distinguished triangle

$$({}_i j_{\mathbf{V},0})_! {}_i j_{\mathbf{V},0}^* \mathcal{L} \longrightarrow \mathcal{L}_{\geq 0} \longrightarrow (i_{\mathbf{V},\geq 1})_* i_{\mathbf{V},\geq 1}^* \mathcal{L} \longrightarrow .$$

That is

$$({}_i j_{\mathbf{V},0})_! \mathcal{L}_0 \longrightarrow \mathcal{L}_{\geq 0} \longrightarrow \mathcal{L}_{\geq 1} \longrightarrow .$$

Hence this theorem is true for  $r = 0$ .

Since  ${}_iE_{\mathbf{V},\geq 2}$  is closed in  $E_{\mathbf{V}}$ , we have the following distinguished triangle

$$(2) \quad ({}_i j_{\mathbf{V},\leq 1})_! {}_i j_{\mathbf{V},\leq 1}^* \mathcal{L}_{\geq 1} \longrightarrow \mathcal{L}_{\geq 1} \longrightarrow (i_{\mathbf{V},\geq 2})_* i_{\mathbf{V},\geq 2}^* \mathcal{L}_{\geq 1} \longrightarrow .$$

Consider the following distinguished triangle

$$({}_i j_{\mathbf{V},\leq 0})_! {}_i j_{\mathbf{V},\leq 0}^* \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{\geq 1} \longrightarrow .$$

Applying the functor  $i_{\mathbf{V},\geq 2}^*$  to this distinguished triangle, we have  $i_{\mathbf{V},\geq 2}^* \mathcal{L}_{\geq 1} = i_{\mathbf{V},\geq 2}^* \mathcal{L}$ .

Hence

$$(i_{\mathbf{V},\geq 2})_* i_{\mathbf{V},\geq 2}^* \mathcal{L}_{\geq 1} = (i_{\mathbf{V},\geq 2})_* i_{\mathbf{V},\geq 2}^* \mathcal{L} = \mathcal{L}_{\geq 2}.$$

Applying the functor  ${}_i j_{\mathbf{V},1}^*$  to this distinguished triangle, we have

$$(3) \quad {}_i j_{\mathbf{V},1}^* \mathcal{L}_{\geq 1} = {}_i j_{\mathbf{V},1}^* \mathcal{L}.$$

The complex  $ij_{\mathbf{V}, \leq 1}^* \mathcal{L}_{\geq 1}$  is a complex on  ${}_i E_{\mathbf{V}, \leq 1}$  and the support is in  ${}_i E_{\mathbf{V}, 1}$ . Hence  $ij_{\mathbf{V}, \leq 1}^* \mathcal{L}_{\geq 1} = (\hat{ij}_{\mathbf{V}, 1})! ij_{\mathbf{V}, 1}^* \mathcal{L}_{\geq 1}$ , where  $\hat{ij}_{\mathbf{V}, 1}$  is the embedding from  ${}_i E_{\mathbf{V}, 1}$  to  ${}_i E_{\mathbf{V}, \leq 1}$ . Then

$$(ij_{\mathbf{V}, \leq 1})! ij_{\mathbf{V}, \leq 1}^* \mathcal{L}_{\geq 1} = (ij_{\mathbf{V}, \leq 1})! (\hat{ij}_{\mathbf{V}, 1})! ij_{\mathbf{V}, 1}^* \mathcal{L}_{\geq 1} = (ij_{\mathbf{V}, 1})! ij_{\mathbf{V}, 1}^* \mathcal{L}_{\geq 1}.$$

By (3), we have

$$(ij_{\mathbf{V}, \leq 1})! ij_{\mathbf{V}, \leq 1}^* \mathcal{L}_{\geq 1} = (ij_{\mathbf{V}, 1})! ij_{\mathbf{V}, 1}^* \mathcal{L} = (ij_{\mathbf{V}, 1})! \mathcal{L}_1.$$

Hence the distinguished triangle (2) can be rewrote as

$$(ij_{\mathbf{V}, 1})! \mathcal{L}_1 \longrightarrow \mathcal{L}_{\geq 1} \longrightarrow \mathcal{L}_{\geq 2} \longrightarrow ,$$

and this theorem is true for  $r = 1$ .

Assume that this theorem is true for  $r = k - 1$ .

Since  ${}_i E_{\mathbf{V}, \geq k+1}$  is closed in  $E_{\mathbf{V}}$ , we have the following distinguished triangle

$$(4) \quad (ij_{\mathbf{V}, \leq k})! ij_{\mathbf{V}, \leq k}^* \mathcal{L}_{\geq k} \longrightarrow \mathcal{L}_{\geq k} \longrightarrow (i_{\mathbf{V}, \geq k+1})_* i_{\mathbf{V}, \geq k+1}^* \mathcal{L}_{\geq k} \longrightarrow .$$

Consider the following distinguished triangle

$$(ij_{\mathbf{V}, \leq k-1})! ij_{\mathbf{V}, \leq k-1}^* \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_{\geq k} \longrightarrow .$$

Applying the functor  $i_{\mathbf{V}, \geq k+1}^*$  to this distinguished triangle, we have  $i_{\mathbf{V}, \geq k+1}^* \mathcal{L}_{\geq k} = i_{\mathbf{V}, \geq k+1}^* \mathcal{L}$ . Hence

$$(i_{\mathbf{V}, \geq k+1})_* i_{\mathbf{V}, \geq k+1}^* \mathcal{L}_{\geq k} = (i_{\mathbf{V}, \geq k+1})_* i_{\mathbf{V}, \geq k+1}^* \mathcal{L} = \mathcal{L}_{\geq k+1}.$$

Applying the functor  $ij_{\mathbf{V}, k}^*$  to this distinguished triangle, we have

$$(5) \quad ij_{\mathbf{V}, k}^* \mathcal{L}_{\geq k} = ij_{\mathbf{V}, k}^* \mathcal{L}.$$

The complex  $ij_{\mathbf{V}, \leq k}^* \mathcal{L}_{\geq k}$  is a complex on  ${}_i E_{\mathbf{V}, \leq k}$  and the support is in  ${}_i E_{\mathbf{V}, k}$ . Hence  $ij_{\mathbf{V}, \leq k}^* \mathcal{L}_{\geq k} = (\hat{ij}_{\mathbf{V}, k})! ij_{\mathbf{V}, k}^* \mathcal{L}_{\geq k}$ , where  $\hat{ij}_{\mathbf{V}, k}$  is the embedding from  ${}_i E_{\mathbf{V}, k}$  to  ${}_i E_{\mathbf{V}, \leq k}$ . Then

$$(ij_{\mathbf{V}, \leq k})! ij_{\mathbf{V}, \leq k}^* \mathcal{L}_{\geq k} = (ij_{\mathbf{V}, \leq k})! (\hat{ij}_{\mathbf{V}, k})! ij_{\mathbf{V}, k}^* \mathcal{L}_{\geq k} = (ij_{\mathbf{V}, k})! ij_{\mathbf{V}, k}^* \mathcal{L}_{\geq k}.$$

By (5), we have

$$(ij_{\mathbf{V}, \leq k})! ij_{\mathbf{V}, \leq k}^* \mathcal{L}_{\geq k} = (ij_{\mathbf{V}, k})! ij_{\mathbf{V}, k}^* \mathcal{L} = (ij_{\mathbf{V}, k})! \mathcal{L}_k.$$

Hence the distinguished triangle (4) can be rewrote as

$$(ij_{\mathbf{V}, k})! \mathcal{L}_k \longrightarrow \mathcal{L}_{\geq k} \longrightarrow \mathcal{L}_{\geq k+1} \longrightarrow ,$$

and this theorem is true for  $r = k$ .

Hence, we have proved this theorem.  $\square$

**Lemma 5.3.** *Fix an  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  and  $0 \leq r \in \mathbb{Z}$ . For any  $\mathcal{L} \in {}_i \mathcal{Q}_{\mathbf{V}, r}$ , we have  $[(ij_{\mathbf{V}, r})! (\mathcal{L})] \in K(\mathcal{Q}_{\mathbf{V}})$ .*

For the proof of this lemma, we need to review Lusztig's constructions of Hall algebras via functions ([13]). For any  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  and  $1 \leq n \in \mathbb{Z}$ , let  $E_{\mathbf{V}}^{F^n}$  and  $G_{\mathbf{V}}^{F^n}$  be the sets consisting of the  $F^n$ -fixed points in  $E_{\mathbf{V}}$  and  $G_{\mathbf{V}}$  respectively, where  $F$  is the Frobenius morphism.

Lusztig defined  $\underline{\mathcal{F}}_{\mathbf{V}}^n$  as the set of all  $G_{\mathbf{V}}^{F^n}$ -invariant  $\overline{\mathbb{Q}}_l$ -functions on  $E_{\mathbf{V}}^{F^n}$  and we can give a multiplication on  $\underline{\mathcal{F}}^n = \bigoplus_{\nu \in \mathbb{N}^I} \underline{\mathcal{F}}_{\mathbf{V}}^n$  to obtain the Hall algebra. For any  $i \in I$ , let  $\mathbf{V}_{ti}$  be the  $I$ -graded  $\mathbb{K}$ -vector space with dimension vector  $ti$  and  $f_i$  be the constant function on  $E_{\mathbf{V}_i}^{F^n}$  with value 1. Denote by  $\mathcal{F}^n$  the composition subalgebra of  $\underline{\mathcal{F}}^n$  generated by  $f_i$  and  $\mathcal{F}_{\mathbf{V}}^n = \underline{\mathcal{F}}_{\mathbf{V}}^n \cap \mathcal{F}^n$ .

For any  $\mathcal{L} \in \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$ , there is a function  $\chi_{\mathcal{L}}^n : E_{\mathbf{V}}^{F^n} \rightarrow \overline{\mathbb{Q}}_l$  (Section III.12 in [8]). Hence, we have the trace map

$$\begin{aligned} \chi^n : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) &\rightarrow \underline{\mathcal{F}}_{\mathbf{V}}^n \\ \mathcal{L} &\mapsto \chi_{\mathcal{L}}^n. \end{aligned}$$

Lusztig proved that  $\chi^n(\mathcal{Q}_{\mathbf{V}}) = \mathcal{F}_{\mathbf{V}}^n$  in [13].

*The proof of Lemma 5.3.* Similarly to the definition of  $\chi^n$ , we can get a function  $f_n = {}_r\chi_{\mathcal{L}}^n$  on  ${}_iE_{\mathbf{V},r}^{F^n}$  for any  $\mathcal{L} \in {}_i\mathcal{Q}_{\mathbf{V},r}$  and  $n \in \mathbb{Z}_{\geq 1}$ . The function  $f_n$  can be viewed as a function on  $E_{\mathbf{V}}^{F^n}$  and  $\chi_{(ij_{\mathbf{V},r})!\mathcal{L}}^n = f_n$ .

By Lemma 5.1, there exists a complex  $\hat{\mathcal{L}} \in \mathcal{Q}_{\mathbf{V}}$  such that  ${}_ij_{\mathbf{V},r}^*\hat{\mathcal{L}} = \mathcal{L}$ . Note that the function  $f_n$  is the restriction of  $\chi_{\hat{\mathcal{L}}}^n$  on  ${}_iE_{\mathbf{V},r}^{F^n}$ . Since  $\chi_{\hat{\mathcal{L}}}^n \in \mathcal{F}_{\mathbf{V}}^n$ , we have  $f_n \in \mathcal{F}_{\mathbf{V}}^n$ .

By Theorem III.12.1 in [8], there is an injective map

$$\prod_{n \in \mathbb{Z}_{\geq 1}} \chi^n : K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})) \rightarrow \prod_{n \in \mathbb{Z}_{\geq 1}} \underline{\mathcal{F}}_{\mathbf{V}}^n.$$

Hence, we have  $[(ij_{\mathbf{V},r})!\mathcal{L}] \in K(\mathcal{Q}_{\mathbf{V}})$ . □

Hence, we get an  $\mathcal{A}$ -linear map  $({}_ij_r)!: K({}_i\mathcal{Q}_r) \rightarrow K(\mathcal{Q})$ .

**Theorem 5.4.** *For any  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  with  $\dim V_i = s$ , we have the following isomorphism of  $\mathcal{A}$ -modules*

$$\begin{aligned} {}_i\Phi_{\mathbf{V}} : K(\mathcal{Q}_{\mathbf{V}}) &\rightarrow \bigoplus_r K({}_i\mathcal{Q}_{\mathbf{V},r}) \\ [\mathcal{L}] &\mapsto ([\mathcal{L}_0], [\mathcal{L}_1], \dots, [\mathcal{L}_s]) \end{aligned}$$

where  $\mathcal{L}_r = {}_ij_{\mathbf{V},r}^*\mathcal{L}$ .

*Proof.* By Lemma 5.1, this map is well-defined. Consider the following map

$$\begin{aligned} {}_i\Psi_{\mathbf{V}} : \bigoplus_r K({}_i\mathcal{Q}_{\mathbf{V},r}) &\rightarrow K(\mathcal{Q}_{\mathbf{V}}) \\ ([\mathcal{L}_0], [\mathcal{L}_1], \dots, [\mathcal{L}_s]) &\mapsto [\bigoplus_r ({}_ij_{\mathbf{V},r})!\mathcal{L}_r]. \end{aligned}$$

By Theorem 5.2, we have  ${}_i\Psi_{\mathbf{V}} \circ {}_i\Phi_{\mathbf{V}}([\mathcal{L}]) = [\mathcal{L}]$ . Hence the map  ${}_i\Phi_{\mathbf{V}}$  is injective. By Lemma 5.3, the map  ${}_i\Phi_{\mathbf{V}}$  is also surjective. □

**Theorem 5.5.** *For any  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  and integer  $0 \leq r \leq \dim V_i$ , fix an  $I$ -graded vector space  $\mathbf{W}$  with  $\dim W_i = \dim V_i - r$  and  $\dim W_j = \dim V_j$  for all  $j \neq i$ . Then we have the following isomorphism of  $\mathcal{A}$ -modules*

$$\begin{aligned} {}_i\mathrm{ind}_{\mathbf{W}}^{\mathbf{V}} : K({}_i\mathcal{Q}_{\mathbf{W},0}) &\rightarrow K({}_i\mathcal{Q}_{\mathbf{V},r}) \\ [\mathcal{L}] &\mapsto [{}_ij_{\mathbf{V},r}^*(\mathrm{Ind}_{\mathbf{V}_{ri}}^{\mathbf{V}}(\mathcal{L}_{ri}, ({}_ij_{\mathbf{W},0})!\mathcal{L}))], \end{aligned}$$

where  $\mathcal{L}_{ri}$  is the constant sheaf on  $E_{\mathbf{V}_{ri}}$ .

*Proof.* By Lemma 5.3, this map is well-defined. Consider the following map

$$\begin{aligned} {}_i\mathrm{res}_{\mathbf{W}}^{\mathbf{V}} : K({}_i\mathcal{Q}_{\mathbf{V},r}) &\rightarrow K({}_i\mathcal{Q}_{\mathbf{W},0}) \\ [\mathcal{L}] &\mapsto [{}_ij_{\mathbf{W},0}^*(\mathcal{L}')], \end{aligned}$$

where  $\mathrm{Res}_{\mathbf{V}_{ri}\mathbf{W}}^{\mathbf{V}}(({}_ij_{\mathbf{V},r})!\mathcal{L}) = \mathcal{L}_{ri} \otimes \mathcal{L}'$ . By definition, we have  ${}_i\mathrm{res}_{\mathbf{W}}^{\mathbf{V}} \circ {}_i\mathrm{ind}_{\mathbf{W}}^{\mathbf{V}} = \mathrm{id}$  and  ${}_i\mathrm{ind}_{\mathbf{W}}^{\mathbf{V}} \circ {}_i\mathrm{res}_{\mathbf{W}}^{\mathbf{V}} = \mathrm{id}$ .  $\square$

By Theorem 5.4 and 5.5, we have the following decomposition of  $K(\mathcal{Q})$ .

**Theorem 5.6.** *The  $\mathcal{A}$ -module  $K(\mathcal{Q})$  has the following direct sum decomposition*

$$K(\mathcal{Q}) = \bigoplus_{r \geq 0} [\mathcal{L}_{ri}] * ({}_ij_0)!(K({}_i\mathcal{Q}_0)).$$

**Proposition 5.7** ([21, 22]). *There exists an isomorphism of  $\mathcal{A}$ -algebras  ${}_i\lambda_{0,\mathcal{A}} : K({}_i\mathcal{Q}_0) \rightarrow {}_i\mathbf{f}_{\mathcal{A}}$  such that the following diagram is commutative*

$$\begin{array}{ccccc} K({}_i\mathcal{Q}_0) & \xrightarrow{({}_ij_0)!} & K(\mathcal{Q}) & \xrightarrow{{}_ij_0^*} & K({}_i\mathcal{Q}_0) \\ \downarrow {}_i\lambda_{0,\mathcal{A}} & & \downarrow \lambda_{\mathcal{A}} & & \downarrow {}_i\lambda_{0,\mathcal{A}} \\ {}_i\mathbf{f}_{\mathcal{A}} & \longrightarrow & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}_i\pi_{\mathcal{A}}} & {}_i\mathbf{f}_{\mathcal{A}}. \end{array}$$

Hence, Theorem 5.6 is a geometric interpretation of Theorem 2.2.

5.2. When  $i \in I$  is a source, we have similar constructions. Let  $\mathbf{V}$  be a finite dimensional  $I$ -graded  $\mathbb{K}$ -vector space such that  $\underline{\dim} \mathbf{V} = \nu \in \mathbb{N}I$ . For any  $r \in \mathbb{Z}_{\geq 0}$ , consider a subvariety  ${}^iE_{\mathbf{V},r}$  of  $E_{\mathbf{V}}$

$${}^iE_{\mathbf{V},r} = \{x \in E_{\mathbf{V}} \mid \mathrm{Ker}(\bigoplus_{\rho \in H, s(\rho)=i} x_{\rho}) \text{ has dimension } r \text{ in } V_i\}.$$

Denote by  ${}^ij_{\mathbf{V},r} : E_{\mathbf{V},r} \rightarrow E_{\mathbf{V}}$  be the natural embedding. Let  $\mathcal{D}_{G_{\mathbf{V}}}^b({}^iE_{\mathbf{V},r})$  be the  $G_{\mathbf{V}}$ -equivariant bounded derived category of  $\overline{\mathbb{Q}}_l$ -constructible complexes on  ${}^iE_{\mathbf{V},r}$ . Naturally, we have two functors  $({}_ij_{\mathbf{V},r})! : \mathcal{D}_{G_{\mathbf{V}}}({}^iE_{\mathbf{V},r}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$  and  ${}^ij_{\mathbf{V},r}^* : \mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}({}^iE_{\mathbf{V},r})$ .

Similarly, we can define  ${}^i\mathcal{P}_{\mathbf{V},r}$ ,  ${}^i\mathcal{Q}_{\mathbf{V},r}$ ,  $K({}_i\mathcal{Q}_{\mathbf{V},r})$ ,  $K({}_i\mathcal{Q}_r)$ ,  $({}_ij_r)!$  and  ${}^ij_r^*$ . We also have the following results.

**Theorem 5.8.** *For any  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  with  $\dim V_i = s$ , we have the following isomorphism*

$$\begin{aligned} {}^i\Phi_{\mathbf{V}} : K(\mathcal{Q}_{\mathbf{V}}) &\rightarrow \bigoplus_r K({}^i\mathcal{Q}_{\mathbf{V},r}) \\ [\mathcal{L}] &\mapsto ([\mathcal{L}_0], [\mathcal{L}_1], \dots, [\mathcal{L}_s]), \end{aligned}$$

where  $\mathcal{L}_r = {}^i j_{\mathbf{V},r}^* \mathcal{L}$ . The inverse of  ${}^i\Phi_{\mathbf{V}}$  is

$$\begin{aligned} {}^i\Psi_{\mathbf{V}} : \bigoplus_r K({}^i\mathcal{Q}_{\mathbf{V},r}) &\rightarrow K(\mathcal{D}_{G_{\mathbf{V}}}(E_{\mathbf{V}})) \\ ([\mathcal{L}_0], [\mathcal{L}_1], \dots, [\mathcal{L}_s]) &\mapsto [\bigoplus_r ({}^i j_{\mathbf{V},r})! \mathcal{L}_r]. \end{aligned}$$

**Theorem 5.9.** *For any  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  and integer  $0 \leq r \leq \dim V_i$ , fix an  $I$ -graded vector space  $\mathbf{W}$  with  $\dim W_i = \dim V_i - r$  and  $\dim W_j = \dim V_j$  for all  $j \neq i$ . Then we have the following isomorphism*

$$\begin{aligned} {}^i\text{ind}_{\mathbf{W}}^{\mathbf{V}} : K({}^i\mathcal{Q}_{\mathbf{W},0}) &\rightarrow K({}^i\mathcal{Q}_{\mathbf{V},r}) \\ [\mathcal{L}] &\mapsto [{}^i j_{\mathbf{V},r}^* (\text{Ind}_{\mathbf{W}\mathbf{V}_{ri}}^{\mathbf{V}} (({}^i j_{\mathbf{W},0})! \mathcal{L}, \mathcal{L}_{ri}))], \end{aligned}$$

whose inverse is

$$\begin{aligned} {}^i\text{res}_{\mathbf{W}}^{\mathbf{V}} : K({}^i\mathcal{Q}_{\mathbf{V},r}) &\rightarrow K({}^i\mathcal{Q}_{\mathbf{W},0}) \\ [\mathcal{L}] &\mapsto [{}^i j_{\mathbf{W},0}^* (\mathcal{L}')], \end{aligned}$$

where  $\text{Res}_{\mathbf{W}\mathbf{V}_{ri}}^{\mathbf{V}} (({}^i j_{\mathbf{V},r})! \mathcal{L}) = \mathcal{L}' \otimes \mathcal{L}_{ri}$ .

**Theorem 5.10.** *The  $\mathcal{A}$ -module  $K(\mathcal{Q})$  has the following direct sum decomposition*

$$K(\mathcal{Q}) = \bigoplus_{r \geq 0} ({}^i j_0)! (K({}^i\mathcal{Q}_0)) * [\mathcal{L}_{ri}].$$

**Proposition 5.11** ([21, 22]). *There exists an isomorphism of  $\mathcal{A}$ -algebras  ${}^i\lambda_{0,\mathcal{A}} : K({}^i\mathcal{Q}_0) \rightarrow {}^i\mathbf{f}_{\mathcal{A}}$  such that the following diagram is commutative*

$$\begin{array}{ccccc} K({}^i\mathcal{Q}_0) & \xrightarrow{({}^i j_0)!} & K(\mathcal{Q}) & \xrightarrow{{}^i j_0^*} & K({}^i\mathcal{Q}_0) \\ \downarrow {}^i\lambda_{0,\mathcal{A}} & & \downarrow \lambda_{\mathcal{A}} & & \downarrow {}^i\lambda_{0,\mathcal{A}} \\ {}^i\mathbf{f}_{\mathcal{A}} & \xrightarrow{\quad} & \mathbf{f}_{\mathcal{A}} & \xrightarrow{{}^i\pi_{\mathcal{A}}} & {}^i\mathbf{f}_{\mathcal{A}}. \end{array}$$

## 6. GEOMETRIC REALIZATION OF LUSZTIG'S SYMMETRY $T_i : \mathbf{U} \rightarrow \mathbf{U}$

In this section, we shall recall the geometric realization of  $T_i : {}^i\mathbf{f} \rightarrow {}^i\mathbf{f}$  in [21, 22]. By using this geometric realization and the structure of  $K(\mathcal{Q})$  in last section, we shall give a geometric realization of Lusztig's symmetry  $T_i : \mathbf{U} \rightarrow \mathbf{U}$ .

6.1. Assume that  $i$  is a sink of  $Q = (I, H, s, t)$ . So  $i$  is a source of  $Q' = \sigma_i Q = (I, H', s, t)$ , where  $\sigma_i Q$  is the quiver by reversing the directions of all arrows in  $Q$  containing  $i$ . For any  $\nu, \nu' \in \mathbb{N}I$  such that  $\nu' = s_i \nu$  and  $I$ -graded  $\mathbb{K}$ -vector spaces  $\mathbf{V}, \mathbf{V}'$  such that  $\underline{\dim} \mathbf{V} = \nu$ ,  $\underline{\dim} \mathbf{V}' = \nu'$ , there exists a functor ([21, 22])

$$\tilde{\omega}_i : {}^i \mathcal{Q}_{\mathbf{V},0} \rightarrow {}^i \mathcal{Q}_{\mathbf{V}',0},$$

which induces the following map

$$\tilde{\omega}_i : K({}^i \mathcal{Q}_0) \rightarrow K({}^i \mathcal{Q}_0).$$

**Theorem 6.1** ([21, 22]). *We have the following commutative diagram*

$$\begin{array}{ccc} K({}^i \mathcal{Q}_0) & \xrightarrow{\tilde{\omega}_i} & K({}^i \mathcal{Q}_0) \\ \downarrow {}^i \lambda_{0,\mathcal{A}} & & \downarrow {}^i \lambda_{0,\mathcal{A}} \\ {}^i \mathbf{f}_{\mathcal{A}} & \xrightarrow{T_i} & {}^i \mathbf{f}_{\mathcal{A}}. \end{array}$$

6.2. By the triangular decomposition (1) of  $DK(\mathcal{Q})(Q)$ , the following set is an  $\mathcal{A}$ -basis of  $DK(\mathcal{Q})(Q)$

$$\{[\mathcal{L}]^- \mathbf{k}_\mu [\mathcal{L}']^+\},$$

where  $\mathcal{L} \in \mathcal{P}_{\mathbf{V}}$  for some  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  with  $\underline{\dim} \mathbf{V} = \nu \in \mathbb{N}I$ ,  $\mathcal{L}' \in \mathcal{P}_{\mathbf{V}'}$  for some  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}'$  with  $\underline{\dim} \mathbf{V}' = \nu' \in \mathbb{N}I$  and  $\mu \in P^\vee$ .

By Theorem 5.6, we have

$$[\mathcal{L}] = \sum_{r \geq 0} [\mathcal{L}_{ri}] * (i j_0)! [\mathcal{L}_r]$$

and

$$[\mathcal{L}'] = \sum_{r \geq 0} [\mathcal{L}_{ri}] * (i j_0)! [\mathcal{L}'_r],$$

where  $[\mathcal{L}_r] = {}_i \text{res}_{\mathbf{W}_r}^{\mathbf{V}} [i j_{\mathbf{V},r}^* \mathcal{L}]$ ,  $[\mathcal{L}'_r] = {}_i \text{res}_{\mathbf{W}_r}^{\mathbf{V}'} [i j_{\mathbf{V}',r}^* \mathcal{L}']$  and  $\mathbf{W}_r$  is an  $I$ -graded  $\mathbb{K}$ -vector space with  $\dim W_i = \dim V_i - r$  and  $\dim W_j = \dim V_j$  for all  $j \neq i$ .

Define

$$\tilde{T}_i([\mathcal{L}]^-) = \sum_{r \geq 0} v^{(\nu, ri)} \mathbf{k}_{-r h_i} [\mathcal{L}_{ti}]^+ (i j_0)! [\tilde{\omega}_i(\mathcal{L}_r)]^-,$$

$$\tilde{T}_i([\mathcal{L}']^+) = \sum_{r \geq 0} v^{(\nu, ri)} \mathbf{k}_{r h_i} [\mathcal{L}_{ti}]^- (i j_0)! [\tilde{\omega}_i(\mathcal{L}'_r)]^+$$

and

$$\tilde{T}_i(\mathbf{k}_\mu) = \mathbf{k}_{s_i(\mu)}.$$

Hence, we get a map

$$\tilde{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q').$$

**Proposition 6.2.** *The map  $\tilde{T}_i$  is a bijection.*



For the proof of this proposition, we should construct the inverse of  $\tilde{T}_i$ . The algebra  $DK(\mathcal{Q})(Q')$  also has the following  $\mathcal{A}$ -basis

$$\{[\mathcal{L}]^{-}\mathbf{k}_\mu[\mathcal{L}']^{+}\},$$

where  $\mathcal{L} \in \mathcal{P}_{\mathbf{V}}$  for some  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}$  with  $\underline{\dim} \mathbf{V} = \nu \in \mathbb{N}I$ ,  $\mathcal{L}' \in \mathcal{P}_{\mathbf{V}'}$  for some  $I$ -graded  $\mathbb{K}$ -vector space  $\mathbf{V}'$  with  $\underline{\dim} \mathbf{V}' = \nu' \in \mathbb{N}I$  and  $\mu \in P^\vee$ .

By Theorem 5.10, we have

$$[\mathcal{L}] = \sum_{r \geq 0} ({}^i j_0)! [\mathcal{L}_r] * [\mathcal{L}_{ri}]$$

and

$$[\mathcal{L}'] = \sum_{r \geq 0} ({}^i j_0)! [\mathcal{L}'_r] * [\mathcal{L}_{ri}],$$

where  $[\mathcal{L}_r] = {}^i \text{res}_{\mathbf{W}_r}^{\mathbf{V}} [{}^i j_{\mathbf{V},r}^* \mathcal{L}]$ ,  $[\mathcal{L}'_r] = {}^i \text{res}_{\mathbf{W}_r}^{\mathbf{V}'} [{}^i j_{\mathbf{V}',r}^* \mathcal{L}']$ .

Define

$$\begin{aligned} \tilde{T}'_i([\mathcal{L}]^{-}) &= \sum_{r \geq 0} v^{\langle ri, \nu \rangle} ({}^i j_0)! [\tilde{\omega}_i^{-1}(\mathcal{L}_r)]^{-} [\mathcal{L}_{ti}]^{+} \mathbf{k}_{rh_i}, \\ \tilde{T}'_i([\mathcal{L}']^{+}) &= \sum_{r \geq 0} v^{\langle ri, \nu' \rangle} ({}^i j_0)! [\tilde{\omega}_i^{-1}(\mathcal{L}'_r)]^{+} [\mathcal{L}_{ti}]^{-} \mathbf{k}_{-rh_i} \end{aligned}$$

and

$$\tilde{T}'_i(\mathbf{k}_\mu) = \mathbf{k}_{s_i(\mu)}.$$

Hence, we get a map

$$\tilde{T}'_i : DK(\mathcal{Q})(Q') \rightarrow DK(\mathcal{Q})(Q).$$

By the definitions of  $\tilde{T}_i$  and  $\tilde{T}'_i$ , the map  $\tilde{T}'_i$  is the inverse of  $\tilde{T}_i$ . Hence, we have proved Proposition 6.2.

The following theorem is the main result in this paper.

**Theorem 6.3.** *The map  $\tilde{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q')$  is an isomorphism of Hopf algebras satisfying that the following diagram is commutative*

$$\begin{array}{ccc} DK(\mathcal{Q})(Q) & \xrightarrow{\tilde{T}_i} & DK(\mathcal{Q})(Q') \\ \downarrow \cong & & \downarrow \cong \\ \mathbf{U}_{\mathcal{A}} & \xrightarrow{T_i} & \mathbf{U}_{\mathcal{A}}. \end{array}$$

*Proof.* Consider the  $\mathcal{A}$ -basis  ${}_i \mathbf{B}_0$  of  $K({}_i \mathcal{Q}_0)$ . By Theorem 5.6, the following set is an  $\mathcal{A}$ -basis of  $DK(\mathcal{Q})(Q)$

$$\{[\mathcal{L}_{ri}]^{-} ({}^i j_0)! (b)^{-} \mathbf{k}_\mu [\mathcal{L}_{r'i}]^{+} ({}^i j_0)! (b')^{+} \mid r, r' \in \mathbb{Z}_{\geq 0}, b, b' \in {}_i \mathbf{B}_0, \mu \in P^\vee\}.$$

Similarly, by Theorem 2.2, the following set is an  $\mathcal{A}$ -basis of  $\mathbf{U}_{\mathcal{A}}$

$$\{F_i^{(r)} {}_i \lambda_{0,\mathcal{A}}(b)^{-} K_\mu E_i^{(r')} {}_i \lambda_{0,\mathcal{A}}(b')^{+} \mid r, r' \in \mathbb{Z}_{\geq 0}, b, b' \in {}_i \mathbf{B}_0, \mu \in P^\vee\}.$$

Note that these two basis are identified under the isomorphism  $\lambda_{\mathcal{A}} : DK(\mathcal{Q})(Q) \cong \mathbf{U}_{\mathcal{A}}$ . By Theorem 6.1 and the definition of  $\tilde{T}_i$ , we have the desired commutative

diagram. Since  $T_i : \mathbf{U}_{\mathcal{A}} \rightarrow \mathbf{U}_{\mathcal{A}}$  is an isomorphism of Hopf algebras, so is the map  $\hat{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q')$ .  $\square$

## 7. BRAID GROUP RELATIONS

In this section, we shall consider the braid group relations of Lusztig's symmetries.

7.1. First, we shall recall the Fourier-Deligne transform ([8][14]). Let  $Q = (I, H, s, t)$  be a quiver. Let  $E$  be a subset of  $H$  and denote by  $Q' = \sigma_E Q$  the quiver obtained from  $Q$  by reversing all the arrows in  $E$ . Given  $\nu \in \mathbb{N}I$ , let  $\mathbf{V}$  be an  $I$ -graded  $\mathbb{K}$ -vector space with dimension vector  $\nu$ . The Fourier-Deligne transform is denoted by ([14])

$$\Theta_{Q,Q'} : \mathcal{D}_{G_{\mathbf{V}}}^b(E_{\mathbf{V},Q}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}^b(E_{\mathbf{V},Q'}).$$

**Lemma 7.1** ([14]). *The Fourier-Deligne transform  $\Theta_{Q,Q'} : \mathcal{D}_{G_{\mathbf{V}}}^b(E_{\mathbf{V},Q}) \rightarrow \mathcal{D}_{G_{\mathbf{V}}}^b(E_{\mathbf{V},Q'})$  is an equivalence of triangulated categories and  $\Theta_{Q,Q'}(\mathcal{Q}_{\mathbf{V},Q}) = \mathcal{Q}_{\mathbf{V},Q'}$ .*

By Lemma 7.1, we have the following proposition.

**Proposition 7.2.** *The Fourier-Deligne transform  $\Theta_{Q,Q'} : \mathcal{Q}_{\mathbf{V},Q} \rightarrow \mathcal{Q}_{\mathbf{V},Q'}$  for various dimension vectors induce an isomorphism of Hopf algebras*

$$\Theta_{Q,Q'} : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q')$$

sending  $[\mathcal{L}]^{-\mathbf{k}_{\mu}}[\mathcal{L}']^{+}$  to  $[\Theta_{Q,Q'}(\mathcal{L})]^{-\mathbf{k}_{\mu}}[\Theta_{Q,Q'}(\mathcal{L}')]^{+}$ .

7.2. For any quiver  $Q = (I, H, s, t)$ , choose a subset  $E$  of  $H$  such that  $i$  is a sink of  $\sigma_E Q$ . Consider the quiver  $\sigma_i \sigma_E Q = (I, H', s, t)$  with  $i$  as a source. Choose a subset  $E'$  of  $H'$  such that  $\sigma_{E'} \sigma_i \sigma_E Q = Q$ .

Define

$$\hat{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q)$$

as the composition of

$$K(\mathcal{Q})(Q) \xrightarrow{\Theta_{Q,\sigma_E Q}} K(\mathcal{Q})(\sigma_E Q) \xrightarrow{\hat{T}_i} DK(\mathcal{Q})(\sigma_i \sigma_E Q) \xrightarrow{\Theta_{\sigma_i \sigma_E Q, Q}} K(\mathcal{Q})(Q) .$$

By using Fourier-Deligne transform, Theorem 6.3 can be rewrote as following.

**Theorem 7.3.** *The map  $\hat{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q)$  is an isomorphism of Hopf algebras satisfying that the following diagram is commutative*

$$\begin{array}{ccc} DK(\mathcal{Q})(Q) & \xrightarrow{\hat{T}_i} & DK(\mathcal{Q})(Q) \\ \downarrow \cong & & \downarrow \cong \\ \mathbf{U}_{\mathcal{A}} & \xrightarrow{T_i} & \mathbf{U}_{\mathcal{A}}. \end{array}$$

Since  $T_i : \mathbf{U} \rightarrow \mathbf{U}$  satisfies the braid group relations, the isomorphism  $\hat{T}_i : DK(\mathcal{Q})(Q) \rightarrow DK(\mathcal{Q})(Q)$  also satisfies the braid group relations, that is, we have the following commutative diagrams

$$\begin{array}{ccccc}
 & & DK(\mathcal{Q})(Q) & \xrightarrow{\hat{T}_j} & DK(\mathcal{Q})(Q) \\
 & \nearrow \hat{T}_i & & & \nwarrow \hat{T}_i \\
 DK(\mathcal{Q})(Q) & & & & DK(\mathcal{Q})(Q) \\
 & \searrow \hat{T}_j & & & \nearrow \hat{T}_j \\
 & & DK(\mathcal{Q})(Q) & \xrightarrow{\hat{T}_i} & DK(\mathcal{Q})(Q)
 \end{array}$$

for any  $i \neq j \in I$  such that  $a_{ij} = -1$ , and

$$\begin{array}{ccccc}
 & & DK(\mathcal{Q})(Q) & & \\
 & \nearrow \hat{T}_i & & \nwarrow \hat{T}_j & \\
 DK(\mathcal{Q})(Q) & & & & DK(\mathcal{Q})(Q) \\
 & \searrow \hat{T}_j & & \nearrow \hat{T}_i & \\
 & & DK(\mathcal{Q})(Q) & & 
 \end{array}$$

for any  $i \neq j \in I$  such that  $a_{ij} = 0$ .

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